

Optical phase estimation in the presence of phase-diffusion

Marco G. Genoni,^{1,2} Stefano Olivares,³ and Matteo G. A. Paris^{2,*}

¹*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK*

²*Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*

³*Dipartimento di Fisica, Università degli Studi di Trieste, I-34151 Trieste, Italy*

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The measurement problem for the optical phase has been traditionally attacked for noiseless schemes or in the presence of amplitude or detection noise. Here we address estimation of phase in the presence of phase diffusion and evaluate the ultimate quantum limits to precision for phase-shifted Gaussian states. We look for the optimal detection scheme and derive approximate scaling laws for the quantum Fisher information and the optimal squeezing fraction in terms of the total energy and the amount of noise. We also found that homodyne detection is a nearly optimal detection scheme in the limit of very small and large noise.

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The estimation of the optical phase in quantum mechanics is a longstanding problem with both fundamental and technological implications. The attempts to define a Hermitian phase-operator conjugate to the number operator are the subject of an extensive literature [1] and several experimental protocols to estimate the value of the optical phase have been also proposed and demonstrated, in particular using different quantum strategies and interferometric setups [2–7], which have been shown to beat the standard quantum limit and to attain the Heisenberg limit [8–11]. More recently the ultimate bounds to precision of phase estimation with Gaussian states have been discussed in details [12, 13] using local quantum estimation theory. Squeezed vacuum state has been shown to be the most sensitive for a given average photon number and two adaptive local measurement schemes have been proposed to attain the Heisenberg limit asymptotically.

The estimation of the optical phase, besides the fundamental interest, is also relevant for optical communication scheme where information is encoded in the phase of laser pulses that must travel long distances between the sender and receiver. In such a context the receiver has to decode the information carried by the optical wave-packets which will be unavoidably degraded by different sources of noise, which have to be duly taken into account in the quantum estimation problem. As a matter of fact, only amplitude and/or detection noise have been taken into account in the analysis of quantum phase estimation, e.g. imperfect photodetection in the measurement stage, or amplitude noise in interferometric setups [14–18]. The role of phase-diffusive noise in phase-estimation have been investigated for qubit [19, 20] and in part for condensate systems [21, 22], while no similar analysis have been done for a continuous variable system. Phase-diffusive noise is the most detrimental for phase-estimation since it destroys the off-diagonal elements of the density matrix. Moreover, any quantum state that is unaffected by phase-diffusion, is also invariant under a phase-shift, and thus is totally useless for phase-estimation.

In this letter we address for the first time phase estimation in the presence of phase diffusion, seek for the optimal Gaussian states, and evaluate the ultimate quantum limits to precision

of phase estimation. We also investigate whether the ultimate performances may be achieved with feasible detection scheme and found that homodyne detection is nearly optimal for very small and large amount of noise.

Let us start by a pico review of local quantum estimation theory [23–26]. When a physical parameter is not directly accessible one has to resort to indirect measurements. Let us denote by ϕ the quantity of interest, X the measured observable, and $\chi = (x_1, \dots, x_M)$ the observed sample. The *estimation problem* amounts to find an estimator, that is a map $\hat{\phi} = \hat{\phi}(\chi)$ from the set of the outcomes to the space of parameters. Classically, optimal estimators are those saturating the Cramér-Rao inequality $\text{Var}(\phi) \geq [MF(\phi)]^{-1}$ which bounds from below the variance $\text{Var}(\phi) = E[\hat{\phi}^2] - E[\hat{\phi}]^2$ of any unbiased estimator of the parameter ϕ . In the Cramér-Rao inequality, M is the number of measurements and $F(\phi)$ is the Fisher Information (FI) $F(\phi) = \int dx p(x|\phi) [\partial_\phi \ln p(x|\phi)]^2$ where $p(x|\phi)$ is the conditional probability of obtaining the value x when the parameter has the value ϕ . The quantum analogue of the Cramér-Rao bound is obtained starting from the Born rule $p(x|\phi) = \text{Tr}[\Pi_x \varrho_\phi]$ where $\{\Pi_x\}$ is the probability operator-valued measure (POVM) describing the measurement and ϱ_ϕ the density operator, labeled by the parameter of interest. Upon introducing the Symmetric Logarithmic Derivative (SLD) L_ϕ as the operator satisfying $2\partial_\phi \varrho_\phi = L_\phi \varrho_\phi + \varrho_\phi L_\phi$ one proves that the FI is upper bounded by the Quantum Fisher Information (QFI) [24] $F(\phi) \leq H(\phi) \equiv \text{Tr}[\varrho_\phi L_\phi^2]$. In turn, the ultimate limit to precision is given by the quantum Cramér-Rao bound $\text{Var}(\phi) \geq [MH(\phi)]^{-1}$. The family of states we are going to deal is a unitary one $\varrho_\phi = U_\phi \varrho_0 U_\phi^\dagger = \sum_k \lambda_k |\lambda_k(\phi)\rangle \langle \lambda_k(\phi)|$, where $|\lambda_k(\phi)\rangle = U_\phi |\lambda_k\rangle$ and $U_\phi = \exp\{-i\phi G\}$ describes a phase-shift with the single-mode number operator $G = a^\dagger a$ as the generator. In this case the SLD may be written as $L_\phi = U_\phi L_0 U_\phi^\dagger$, where L_0 is independent on ϕ . The corresponding QFI does not depend on the parameter ϕ , and reads

$$H = \text{Tr}[\varrho_0 L_0^2] = 2 \sum_{n \neq m} \frac{(\lambda_n - \lambda_m)^2}{\lambda_n + \lambda_m} |\langle \lambda_n | G | \lambda_m \rangle|^2 \quad (1)$$

Phase-diffusion for a continuous-variable system is described by the master equation $\dot{\varrho} = \Gamma \mathcal{L}[a^\dagger a] \varrho$ where $\mathcal{L}[O] \varrho = 2O\varrho O^\dagger - O^\dagger O \varrho - \varrho O^\dagger O$. The solution for an initial state $\varrho(0)$ is given by $\varrho(t) = \mathcal{N}_\Delta(\varrho(0)) = \sum_{n,m} e^{-\Delta^2(n-m)^2} \varrho_{n,m}(0) |n\rangle \langle m|$ where $\Delta \equiv \Gamma t$, Γ is the noise amplitude and $\varrho_{n,m}(0) = \langle n | \varrho(0) | m \rangle$. The diagonal elements ϱ are left unchanged and, in turn, energy is conserved, whereas the off-diagonal ones are progressively destroyed.

We assume that phase noise occurs between the application of the phase-shift and the detection of the signal, and address quantum estimation of a phase-shift applied to pure single-mode Gaussian states $|\psi_G\rangle = D(\alpha)S(r)|0\rangle$ where $S(r) = \exp\{(r/2)(a^2 - a^{\dagger 2})\}$ is the squeezing operators, $D(\alpha) = \exp\{\alpha(a^\dagger - a)\}$ the displacement operator, being $r, \alpha \in \mathbb{R}$. The input state is firstly phase-shifted by applying the unitary operator U_ϕ , where ϕ is the unknown phase-shift, and then, before being measured, it undergoes phase-diffusion. Our aim is to determine the ultimate bound to precision for a generic pure Gaussian probe and then look for the optimal one by maximizing the QFI over the state parameters.

The mixed non-Gaussian state that is being measured is given by

$$\varrho_\phi(t) = \mathcal{N}_\Delta(U_\phi |\psi_G\rangle \langle \psi_G| U_\phi^\dagger) = U_\phi \mathcal{N}_\Delta(|\psi_G\rangle \langle \psi_G|) U_\phi^\dagger,$$

where the second equality holds since the superoperator $\mathcal{L}[a^\dagger a]$ and the phase-shift operator U_ϕ commute. Because of this fact our estimation problem corresponds to the case of a unitary family described above, with the input mixed state given by $\mathcal{N}_\Delta(|\psi_G\rangle \langle \psi_G|)$. In order to evaluate the corresponding QFI one writes ϱ_ϕ in its diagonal form $\varrho_\phi = \sum_n \lambda_n |\lambda_n(\phi)\rangle \langle \lambda_n(\phi)| = \sum_n \lambda_n U_\phi |\lambda_n\rangle \langle \lambda_n| U_\phi^\dagger$, where $|\lambda_n(\phi)\rangle$ and $|\lambda_n\rangle$ are respectively the eigenvectors of ϱ_ϕ and of $\mathcal{N}_\Delta(|\psi_G\rangle \langle \psi_G|)$ corresponding to the eigenvalues λ_n , which are in fact left unchanged by the phase-shift operation. By decomposing $|\lambda_n\rangle = \sum_k r_{nk} |k\rangle$ in the Fock basis and by substituting this into the eigenvalues equation $\mathcal{N}_\Delta(|\psi_G\rangle \langle \psi_G|) |\lambda_n\rangle = \lambda_n |\lambda_n\rangle$ we have $\langle n | \psi_G \rangle \langle \psi_G | l \rangle e^{-\Delta^2(n-k)^2} r_{qk} = \lambda_q r_{qn} \forall n$. Moreover, since $a^\dagger a |\lambda_n\rangle = \sum_k k r_{nk} e^{ik\phi} |k\rangle$, we have that: $|\langle \lambda_m | a^\dagger a | \lambda_n \rangle|^2 = |\sum_k k r_{mk} r_{nk}|^2$. After evaluating the QFI using the above formulas one sees that it depends only on the eigenvalues λ_n and on the components of the eigenvectors r_{nk} which, being ϕ a unitary parameter, do not depend on the parameter itself. The explicit values of λ_n and r_{nk} have been obtained by performing numerical diagonalization.

Upon inspecting the solution of the master equation one sees that the vanishing of the off-diagonal matrix elements is governed by the product between Δ^2 and the squared difference between the Fock indices $(n-m)^2$. Besides, for a pure Gaussian state, the presence of non-zero (non-negligible) off-diagonal elements is somehow ruled by the average photon-number $N = \langle a^\dagger a \rangle$ and thus we roughly expect the QFI to somehow depend on the quantity $\xi = N\Delta$. Pure Gaussian states may conveniently be parametrized by the average photon number of photons N and of the corresponding squeezing

fraction β , in formula $N = \sinh^2 r + |\alpha|^2$ and $\beta = \sinh^2 r / N$, and thus the QFI will be function of the three parameters N , β and Δ .

We start our analysis by evaluating the QFI at fixed noise Δ . We consider four values of the maximum energy $N_{\max} = \langle a^\dagger a \rangle_{\max} = \{10, 15, 20, 30\}$ (with 10 steps on intermediate energies N) and different values of the noise parameter Δ . The values of Δ are chosen such that we can find points corresponding to fixed values of ξ . The curves are built by looking for the optimal pure Gaussian state, i.e. maximizing the QFI as a function of the squeezing fraction β , for any fixed value of the energy N and of the noise parameter Δ .

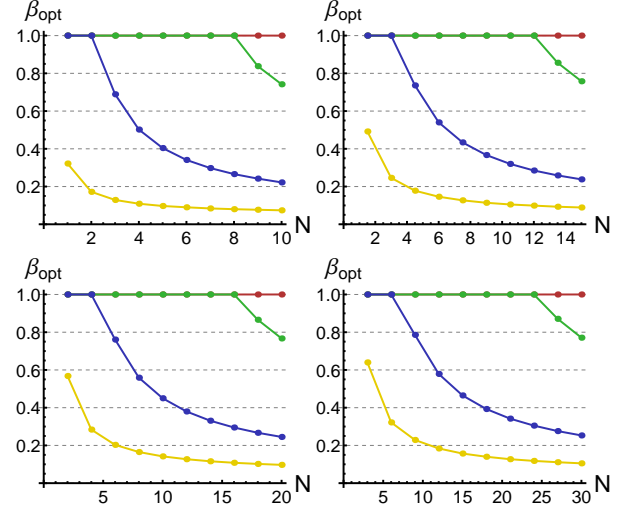


FIG. 1: (Color online) Optimal squeezing fraction β as a function of the average photon number N and for different values of Δ^2 . (Top left): from top to bottom $\Delta^2 = \{4.5 \times 10^{-5}, 4.5 \times 10^{-4}, 4.5 \times 10^{-3}, 4.5 \times 10^{-2}\}$. (Top right): from top to bottom $\Delta^2 = \{2.0 \times 10^{-5}, 2.0 \times 10^{-4}, 2.0 \times 10^{-3}, 2.0 \times 10^{-2}\}$. (Bottom left): from top to bottom $\Delta^2 = \{1.125 \times 10^{-5}, 1.125 \times 10^{-4}, 1.125 \times 10^{-3}, 1.125 \times 10^{-2}\}$. (Bottom right): from top to bottom $\Delta^2 = \{5.0 \times 10^{-6}, 5.0 \times 10^{-5}, 5.0 \times 10^{-4}, 5.0 \times 10^{-3}\}$.

The values of the optimal squeezing fraction $\beta_{\text{opt}} = \beta_{\text{opt}}(N, \Delta)$ and of the corresponding QFI $H(N, \beta_{\text{opt}}, \Delta)$ have been numerically evaluated and are reported in Fig. 1 and Fig. 2 respectively. As we can see in Fig. 1, for a low level noise the squeezing fraction is almost equal to one. In particular, in each plot, for the lowest value of Δ , we obtain $\beta_{\text{opt}}(N, \Delta) = 1$ and thus the optimal probe state is the squeezed vacuum state, as it happens in the noiseless case [12]. As far as the noise Δ increases the squeezing fraction decreases as a function of the average number of photons. This means that for increasing values of the noise and of the energy, it is more convenient to employ the energy in increasing the coherent amplitude rather than the squeezing of the probe. Let us now focus on the behavior of the QFI $H(N, \beta_{\text{opt}}, \Delta)$. In the left panel of Fig. 2, we report the typical behavior of the QFI as a function of N and for different values of Δ . The QFI increases by increasing the average photon number N , and decreases with the noise parameter Δ . For

the lowest value of Δ , we also observe that the noiseless limit $H(N, \beta = 1, \Delta = 0) = 8(N^2 + N)$ [12] is approached, at least for N not too large.

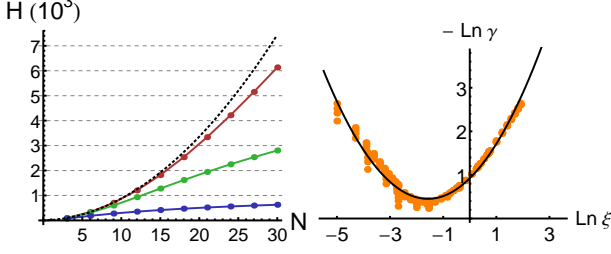


FIG. 2: (Color online) Left panel: QFI of optimized pure input Gaussian states as a function of the average photon number N and for different values of the noise parameter Δ ; from top to bottom $\Delta^2 = \{5.0 \times 10^{-6}, 5.0 \times 10^{-5}, 5.0 \times 10^{-4}\}$. The black dotted line is the QFI for the noiseless case $H(N, \beta = 1, \Delta = 0) = 8(N^2 + N)$. Right panel: (orange points) $-\ln \gamma(\xi)$ as a function of $\ln \xi$, $\xi \equiv N\Delta$, with $N \leq 30$ and $10^{-3} \leq \Delta \leq 1$. The black curve is a best fit with functional form $\gamma(\xi) \propto \xi^{-b} \exp(-a \ln^2 \xi)$.

As we have already mentioned above, because of the form of the phase-diffusion map, we expect that the product $\xi = N\Delta$ plays a role in the estimation properties. In fact, by exploring a large range of values for N and Δ a scaling law emerges from numerical analysis, which may be written as

$$H(N, \Delta) \simeq k^2 H(N/k, k\Delta). \quad (2)$$

That is, $H(N, \Delta) = N/\Delta \gamma(\xi) = N^2 \gamma(\xi)/\xi = \xi \gamma(\xi)/\Delta^2$ where $0 < \gamma(\xi) < 1$ is a universal function independent on Δ and N . The larger is ξ the more accurate is the scaling law. If we fix ξ the QFI for different pairs of N and Δ have the same value, up to a rescaling by a factor k^2 , where k is the ratio between the two average photon numbers, or equivalently of the two noise parameters. The scaling is illustrated in the right panel of Fig. 2 where we report the quantity $-\ln \gamma(\xi)$ as a function of $\ln \xi$ (orange points) together with a two-parameter fit (black curve) of the form $\gamma(\xi) \propto \xi^{-b} \exp(-a \ln^2 \xi)$, that provides a good representation of data. Using the above results, the quantum Cramér-Rao bound for the precision of an optimal estimator of the phase-shift may be written as $\text{Var}(\phi) \gtrsim \frac{\Delta}{\gamma(\xi)N} = \frac{\xi}{\gamma(\xi)N^2}$. For small values of ξ the quantity $\xi \gamma(\xi)$ is of order of unity and thus Heisenberg limit $\text{Var}(\phi) \sim N^{-2}$ in precision may be achieved [27]. We also found that another scaling law, though less accurate, holds also for the optimal squeezing fraction

$$\beta_{\text{opt}}(N, \Delta) \simeq \beta_{\text{opt}}(N/k, k\Delta). \quad (3)$$

Though based on a physical and mathematical justification, we cannot expect these scaling to be exact due to the non-Gaussianity of the state. However they give a useful and practical receipt to compare and predict phase estimation performances in different regimes of energy and noise. In the left panel of Fig. 3 we show the behavior of the quantum Fisher information at fixed average photon number as a function of

Δ . We notice that the $H(N, \Delta)$ is decreasing exponentially with the phase noise and that higher values of N correspond to higher values of H .

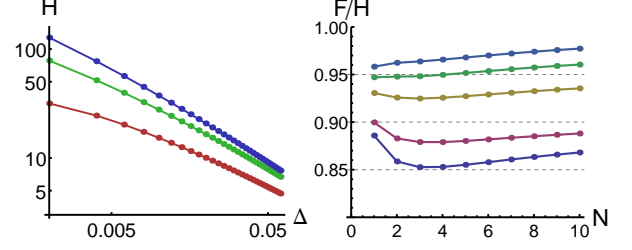


FIG. 3: (Color online) (Left): Log-log plot of the QFI for optimized pure input Gaussian states as a function of the noise parameter Δ for different values of the average photon number. From bottom to top: $N = \{2, 5, 10\}$. (Right): Ratio between the Fisher Information of homodyne detection on coherent states and the corresponding QFI, as a function of the number of photons of the probe states and for different values of Δ . From bottom to top: $\Delta^2 = \{0.5, 1.0, 1.5, 2.0, 5\}$.

In the noiseless case ($\Delta = 0$) homodyne detection performed on input squeezed vacuum states is optimal [12], that is, its Fisher information F is equal to the QFI, $H(N) = 8(N^2 + N)$. A question thus arises on whether this results also holds in the presence of phase diffusion. Our numerical findings shows that this is true for a very small amount of noise, i.e. $\Delta \ll 1$, whereas for increasing Δ the ratio F/H is moving away from unity quite quickly. On the other hand, one can see that for high values of Δ , basically when coherent states are the optimal probe states maximizing the QFI, homodyne detection of the quadrature $X = (a + a^\dagger)/2$ is again nearly optimal, i.e. its Fisher information is again approaching the value of the QFI evaluated in same conditions. In the right panel of Fig. 3 we plot the ratio between the Fisher information of homodyne detection and the corresponding QFI: by increasing the noise Δ the ratio increases towards optimality ($F/H = 1$). This may understood looking at the behavior of quadrature fluctuations $\Delta X_\theta^2 = \langle X_\theta^2 \rangle - \langle X_\theta \rangle^2$ since the smaller is ΔX_θ^2 for a certain quadrature X_θ , the more precise is the estimation of the phase-shift through this quadrature. In Fig. 4, we report a contour plot of $\log \Delta X_\theta^2$ as a function of the squeezing fraction of the input state β and the quadrature phase θ for different values of Δ and of the overall energy N . We see that for low noise, i.e. $\Delta \ll 1$, minimum fluctuations are obtained for the quadrature $\theta = \pi/2$ and for a squeezed vacuum state ($\beta = 1$), whereas after a certain energy-dependent threshold level of noise $\Delta^* \equiv \Delta^*(N)$, we have a jump and the minimum fluctuations are achieved by measuring the X quadrature ($\theta = 0$) on coherent probes ($\beta = 0$). This behavior is different compared to the behavior we have obtained for the QFI, see Fig. 1. There, for intermediate values of Δ , the optimal squeezing fraction decreases monotonically from $\beta = 1$ to $\beta = 0$, whereas here we have only the extreme values. This exactly corresponds to the result discussed above: homodyne detection, as far as we tune

accordingly the measured quadrature, is optimal for very low noise with squeezed vacuum probes ($\beta = 0$), and for large noise with coherent probes ($\beta = 1$), while for intermediate values of Δ homodyne detection is far from optimality. Overall, we have that homodyne detection provides nearly optimal phase estimation in the presence of either very small or large phase diffusion, whereas it is still an open problem to find a feasible measurement attaining the ultimate precision for a generic value of the phase-diffusion noise parameter Δ .

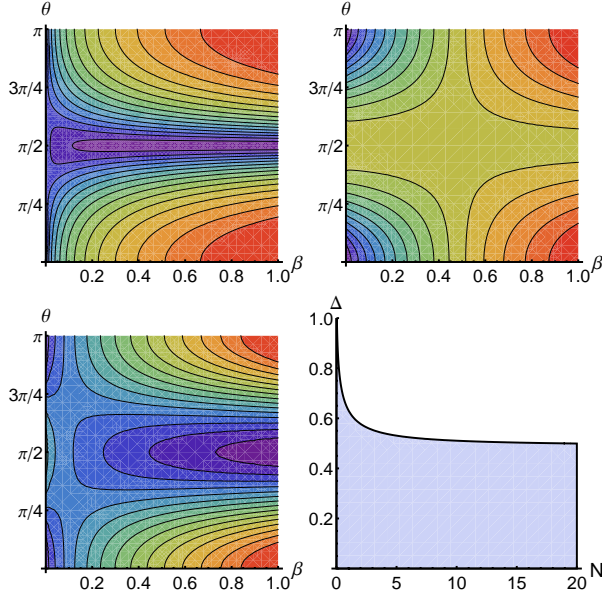


FIG. 4: (Color online) Quadrature fluctuations ΔX_θ^2 as a function of the squeezing fraction of the input state β and of the phase θ for different values of the noise amplitude Δ and the overall energy N . Top left: $N = 10$ and $\Delta = 0.1$; top right: $N = 10$ and $\Delta = 0.6$; bottom left: $N = 0.1$ and $\Delta = 0.1$. Darker regions corresponds to smaller ΔX_θ^2 . The plot in the bottom right panel illustrates the threshold $\Delta^*(N)$ between the two regions where minimum fluctuations are achieved for $\beta = 1, \theta = \pi/2$ (gray area) and $\beta = 0, \theta = 0$ respectively.

In conclusion, we have attacked for the first time the problem of finding the optimal way to estimate a phase-shift in the presence of phase diffusion and we have obtained the ultimate quantum limits to precision for phase-shifted Gaussian states. By an extensive numerical analysis we have obtained an approximate scaling laws for both the quantum Fisher information and the optimal squeezing fraction in terms of the overall total energy and the amount of noise. We also found that homodyne detection is a nearly optimal detection scheme for very small or large noise. Our results goes beyond the traditional analysis of the quantum phase measurement problem and may be relevant for the development of phase-shift keyed optical communication schemes [28].

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* Electronic address: matteo.paris@fisica.unimi.it

- [1] L. Susskind, J. Glogower, *Physics* **1**, 49 (1964); P. Carruthers, M. Nieto, *Rev. Mod. Phys.* **40**, 411 (1968); R. Lynch, *Phys. Rep.* **256**, 367 (1995); R. Tanas, A. Miranowicz, Ts. Gantsog, *Progr. Opt.* **XXXV**, 355 (1996); D. T. Pegg, S. M. Barnett, *J. Mod. Optics*, **44**, 225 (1997).
- [2] M. A. Armen, J. K. Au, J. K. Stockton, A. C. Doherty, and H. Mabuchi, *Phys. Rev. Lett.* **89**, 133602 (2002).
- [3] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, *Nature* **429**, 161 (2004).
- [4] T. Nagata, R. Okamoto, J. L. O'Brien, K. Sasaki, and S. Takeuchi, *Science* **316**, 726 (2007).
- [5] K. J. Resch, K. L. Pagnell, R. Prevedel, A. Gilchrist, G. J. Pryde, J. L. O'Brien, and A. G. White, *Phys. Rev. Lett.* **98**, 223601 (2007).
- [6] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, *Nature* **450**, 393 (2007).
- [7] B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, *New J. Phys.* **11**, 073023 (2009).
- [8] Z. Hradil *Quantum Opt.* **4**, 93 (1992).
- [9] S. L. Braunstein, *Phys. Rev. Lett.* **69**, 3598 (1992).
- [10] V. Giovannetti, S. Lloyd, and L. Maccone, *Science* **306**, 1330 (2004); *Phys. Rev. Lett.* **96**, 010401 (2006).
- [11] F. W. Sun, B. H. Liu, Y. X. Gong, Y. F. Huang, Z. Y. Ou, G. C. Guo, *EPL* **82**, 24001 (2008).
- [12] A. Monras, *Phys. Rev. A* **73**, 033821 (2006).
- [13] S. Olivares, M. G. A. Paris, *J. Phys. B* **42**, 055506 (2009).
- [14] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek and I. A. Walmsley, *Phys. Rev. Lett.* **102**, 040403 (2009); *Phys. Rev. A* **80**, 013825 (2009).
- [15] M. G. A. Paris, *Phys. Lett A* **201**, 132 (1995).
- [16] M. Kacprowicz, R. Demkowicz-Dobrzanski, W. Wasilewski, K. Banaszek, I. A. Walmsley, *Nat. Phot.* **4**, 357 (2010).
- [17] H. Cable, G. A. Durkin, *Phys. Rev. Lett.* **105**, 013603 (2010).
- [18] S. Knysh, V. N. Smelyanskiy and G. A. Durkin, arXiv:1006.1645v3 [quant-ph].
- [19] D. Brivio, S. Cialdi, S. Vezzoli, B. Teklu, M. G. Genoni, S. Olivares, M. G. A. Paris, *Phys. Rev. A* **81**, 012305 (2010).
- [20] B. Teklu, M. G. Genoni, S. Olivares, M. G. A. Paris, *Phys. Scr.* **T140**, 014062 (2010);
- [21] I. Tikhonov, M. G. Moore and A. Vardi, *Phys. Rev. A* **82**, 043624 (2010).
- [22] Y. C. Liu, G. R. Jin, L. You, *Phys. Rev. A* **82**, 045601 (2010).
- [23] J. D. Malley, J. Hornstein, *Stat. Sci.* **8**, 433 (1993).
- [24] S. L. Braunstein and C. M. Caves, *Phys. Rev. Lett.* **72**, 3439 (1994); S. L. Braunstein, C. M. Caves and G. J. Milburn, *Ann. Phys.* **247**, 135 (1996).
- [25] D. C. Brody, L. P. Hughston, *Proc. Roy. Soc. Lond. A* **454**, 2445 (1998); *A* **455**, 1683 (1999).
- [26] M. G. A. Paris, *Int. J. Quant. Inf.* **7**, 125 (2009).
- [27] A. S. Lane, S. Braunstein, C. M. Caves, *Phys. Rev. A* **47**, 1667 (1993).
- [28] L. G. Kazovsky, G. Kalogerakis, W-T. Shaw, *J. Light. Tech.* **24**, 4876 (2006).